# Math 630-102
# Homework #11
# Due date: April 19, 2007

**Group work on h/w assignments is not allowed. No credit is given for results without a solution or an explanation. Late homework is not accepted.**

### Section 5.2

**Problem I.** Any symmetric matrix ($A^T=A$) satisfies the following two properties:

1. It has a full set of *n* linearly independent eigenvectors, and hence can always be diagonalized, even if its eigenvalues are not distinct.

2. Its eigenvectors are mutually orthogonal, and all eigenvalues and eigenvectors are real.

Verify these results for the matrix $A=\begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

### Section 5.3

**Problem II.** *Markov processes* in probability theory provide another practical application of linear difference equations, where a matrix called the Markov matrix specifies the probability of transitions between different "states" of some system. For instance, consider the following simple game:

Initially, a ball is placed in one of two boxes, a yellow box (Y) or a green box (G). A fair die (a cube with faces numbered 1 to 6) is rolled repeatedly, and the result of each roll determines whether the ball will be moved, or stays in place, according to the following rules:

1. If the ball is in yellow box (Y) and the die roll produces 1 or 3, the ball is moved to the green box (G); otherwise it stays in box Y.

2. If the ball is in box G and the die roll produces an even number (2, 4 or 6), the ball is moved to box Y; otherwise it stays in box G.

Let's denote $p_k^Y$ the probability that the ball is in the yellow box after *k* rolls, and denote $p_k^G$ the probability that the ball is in the green box after *k* rolls. It is not hard to see that these probabilities change after each die roll according to the difference equation:

$$\begin{bmatrix} p_{k+1}^Y \\ p_{k+1}^G \end{bmatrix} = \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} p_k^Y \\ p_k^G \end{bmatrix} \rightarrow u_{k+1} = A\, u_k \text{, where } u_k = \begin{bmatrix} p_k^Y \\ p_k^G \end{bmatrix}$$

Note that $A_{21}$=1/3 (the transition probability from Y to G) is the probability for a die roll to show 1 or 3, since 2/6=1/3. The element $A_{12}$=1/2 (the transition probability from G to Y) is the probability that a die roll is an even number.

The matrix in this equation is called the Markov matrix; the sum of elements in each column of the matrix has to equal 1 (the probability to move plus the probability to stay equals 1). One important property of any Markov matrix is that it always has an eigenvalue $\lambda$=1, and all other eigenvalues are less than one.

a) Find the eigenvalues and the eigenvectors of this matrix.
b) Find $p_k^Y$ and $p_k^G$ if the ball is initially in the yellow box ( $p_0^Y$=1, $p_0^G$=0).
c) Can you tell what happens after a great number of game turns, *k*? Do the probabilities approach a steady state? If you place a bet that the ball will be in the yellow box after 40 rounds, what are your chances (odds) of winning?

## Section 5.4

**Problem III.** Solve the following differential equation using the diagonalization *u*(t) = $e^{At}u(0) = Se^{\Lambda t}S^{-1}u(0)$

$$\begin{cases} \dfrac{dx}{dt} = x + 4y \\ \dfrac{dy}{dt} = x + y \end{cases}$$

Use the initial condition *x*(0)=4, *y*(0)=0

**Problem IV.** Suppose that the rabbit population and the wolf population sizes in a certain park are governed by

$$\begin{cases} \dfrac{dr}{dt} = 2r - w \\ \dfrac{dw}{dt} = r + 2w \end{cases}$$

a) From the determinant and the trace of the matrix alone, figure out whether the two population sizes are going to increase, decrease, or stay constant (see p. 271)
b) After a long time, what is the proportion of rabbits to wolves? (hint: you only need the eigenvalues and the eigenvectors).

**Problem V.** What we've learned can be used to solve differential equations of any order, for instance a *second-order* differential equation such as $\frac{d^2y}{dt^2}+a\frac{dy}{dt}+b\,y=c$, where $a$, $b$ and $c$ are arbitrary constants. Consider for instance the following linear system, where we denote $y'=\frac{dy}{dt}$:

$$\frac{d}{dt}\begin{bmatrix} y' \\ y \end{bmatrix}=\begin{bmatrix} 1 & -5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}$$

a) Write down the second-order differential equation described by this matrix expression
b) From the determinant and the trace of the matrix alone, figure out whether $y'$(t) and $y$(t) will increase or decrease or stay the same (see p. 271)

**Problem VI.** Consider the matrix A = $\begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.1 \end{bmatrix}$

Based on the geometric series $\frac{1}{1-x}=1+x+x^2+x^3+\ldots$ we can infer the following expression for the inverse $(I-A)^{-1}$

$$(I-A)^{-1} = I + A + A^2 + A^3 + A^4 + \ldots.$$

a) Calculate the left-hand side and calculate the first three terms ( $I + A + A^2$ ) on the right-hand side. Compare the two results.
b) Diagonalize A and verify that

$$(I-A)^{-1} = S\,(I-\Lambda)^{-1}\,S^{-1}, \text{ where } (I-\Lambda)^{-1} = \begin{bmatrix} (1-\lambda_1)^{-1} & 0 \\ 0 & (1-\lambda_2)^{-1} \end{bmatrix}$$

c) Is this series expansion for $(I-A)^{-1}$ valid if A is some projection matrix? (hint: what is the square of a projection matrix?). What about a permutation matrix?

**Summary: linear differential equations**

- A *linear* differential equation of any order can be put in the form $\frac{d}{dt}u(t) = Au(t)$ , where A is a matrix describing the system, and *t* is the time variable.

- The solution to the above equation is $u(t)=e^{At}u(0)$, where $u(0)$ is the initial state of the system.

- The solution is **always** (whether the matrix is diagonalizable or not) equal to

  $$\boldsymbol{u(t) = e^{At} u(0)}$$

  where the matrix exponential is defined through its Taylor series

  $$\boldsymbol{e^{A} = I + A + A^2/2! + A^3/3! + A^4/4! + \ldots}$$

- If matrix A can be diagonalized, the solution can be written as

  $$\boldsymbol{u(t) = e^{At} u(0) = S e^{\Lambda t} S^{-1} u(0)}$$

  or, equivalently,

  $$\boldsymbol{u_k = S e^{\Lambda t} c} = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + \ldots + c_n e^{\lambda_n t} x_n$$

where $\mathbf{c = S^{-1}u(0)}$ is the vector of coefficients in the expansion of $\mathbf{u(0)}$ as a linear combination of eigenvectors:

$$\boldsymbol{u(0) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = S c}$$

- Note complete analogy with the difference equations: time variable ***k*** becomes ***t***, and $\lambda^{\mathbf{k}}$ becomes $\mathbf{e}^{\lambda t}$

- The stability of the equilibrium u(0)=0 is determined by the real parts of the eigenvalues, and **only in the 2×2 case** is completely specified by **det(A)** = $\boldsymbol{\lambda_1 \cdot \lambda_2}$ and **trace(A)** = $\boldsymbol{\lambda_1 + \lambda_2}$, as follows:

If det(A) is positive, that means that eigenvalues (more accurately, their real parts) are either both positive, and hence the solution "blows up" (unstable), or both negative, in which case the solution decays to zero (stable). Since the trace of the matrix equals the sum of its eigenvalues, the sign of the trace allows to distinguish between these two different outcomes. If det(A) is negative, then one of the eigenvalues has a positive real part, so the solution is unstable unless the initial condition lies along the other, negative eigenvalue.